Corrigenda Random Graphs and Complex Networks. Volume Two

Remco van der Hofstad

In this document, we list corrections to Random Graphs and Complex Networks Volume 2 [1]. The line numbers refer to the original Cambridge University Press edition.

1 Page 35, (1.3.65) : Equation [II, (1.3.65)] should be replaced by

$$\mathbb{P}\big(v^{(m)}_{n+1,j+1} \to v^{(m)}_i \mid \mathrm{PA}^{(m,\delta)}_{n,j}(d)\big) = \frac{D_i(n,j)+\delta}{2m(n-1)+\delta n+j} \qquad \text{for } i \in [n]. \tag{1.1.1}$$

The graph $\mathrm{PA}^{(m,\delta)}_n(d)$ starts at time $n = 2$ with m edges between the first two vertices. This creates further confusion later on as well.
[Thanks to Chunyin Siu for noting this typo.]

2 Page 52, line -4: Here the '(possibly random)' should be removed. Indeed, since we are dealing with *deterministic* graphs, the limit of the proportions of neighborhoods of certain shape will always be deterministic. [Thanks to Serte Donderwinkel for noting this typo.]

3 Page 56, Definition 2.11 should be called 'Local convergence of random graphs' and not 'Local weak convergence of random graphs'', as we also discuss local convergence in probability and almost surely.

4 Page 196. Equation (5.2.31) should read

$$\frac{(2s_1(n)+d_1-2)+\delta s_1(n)}{(2s_1(n)+d_1-2)+\delta s_1(n)+(2s_2(n)+d_2-2)+\delta s_2(n)}, \tag{1.1.2}$$

[Thanks to Haodong Zhu for noting this typo.]

5 Page 199, line 15: The statement should read 'Recall from Section 1.3.5 that the graph starts at time $n = 2$ with two vertices having m edges between them.' See also item 1 above.

6 Page 200, 2 lines before (5.3.2): Y should be replaced by Γ.
[Thanks to Haodong Zhu for noting this typo.]

7 Page 205, (5.3.19): We take $\psi_1 = 1$ here, (5.3.19) is correct for $j \geq 2$.
[Thanks to Haodong Zhu for noting this typo.]

8 Page 206, Remark 5.11: The statement should read 'For $\mathrm{PA}^{(m,\delta)}_n(d)$, we note that $\sum_{v\in[n]} D_v(n) = 2m(n-1)$ and that it has n vertices, since the graph at time $n = 2$ consists of two vertices with m edges between them.' See also item 1 above.

9 Page 208, line 5: The statement should read 'Indeed, recall from Section 1.3.5 that the graph starts at time $n = 2$ with two vertices having m edges between them.' See also item 1 above.

10 Page 208, (5.3.25): The notation is not exactly optimal here, and it would have been better to write $d^{(G)}_{v_j(u)}(u, j-1)$ for the degree of vertex $v_j(u)$ after the first $j-1$ edges of vertex u are attached. Equation (5.3.26), which is one of the key ingredients in this proof, is true though.
[Thanks to Haodong Zhu for noting this typo.]

11 Page 213, (5.4.7): The variable χ'_k should be replaced by χ_k. This also applies the proof below (5.4.7).
[Thanks to Haodong Zhu for noting this typo.]

12 Page 215, (5.4.20): The sum over i should start at $k+1$, not k. This appears three times in (5.4.20), twice in the sentence following (5.4.20), and twice in (5.4.21).
[Thanks to Haodong Zhu for noting this typo.]

13 Page 215, below (5.4.21): The statement should read 'Using that $|\log(1-x)| \leq x/(1-x)$ for all $x \in [0,1]$...' [Thanks to Haodong Zhu for noting this typo.]

14 Page 215, before (5.4.24): The statement should read 'now using that $x \leq |\log(1-x)| \leq x + x^2/(1-x)$ for all $x \in [0,1]$...' Further, (5.4.24) should read

$$0 \leq \sum_{i=k}^{n} \big(\mathbb{E}[-\log(1-\psi_i)] - \mathbb{E}[\psi_i]\big) \leq \sum_{i=k}^{n} \mathbb{E}\Big[\frac{\psi_i^2}{1-\psi_i}\Big] \leq C \sum_{i\geq k} \frac{1}{i^2}, \tag{1.1.3}$$

[Thanks to Haodong Zhu for noting this typo.]

15 Page 215: (5.4.26) should read

$$\log S_k^{(n)} = \chi \log(n/k) + O(1/k), \tag{1.1.4}$$

[Thanks to Haodong Zhu for noting this typo.]

16 Page 221, Proposition 5.22: There is a right parenthesis too many in $(v_w)_{w\in V(\mathbf{t})})$ in 'Fix $\bar{\mathbf{t}}$ such that $(v_w)_{w\in V(\mathbf{t})})$ are all distinct, with the oldest vertex having age at least εn.'

17 Page 222: The 'no-further-edge' factor

$$\prod_{v\in V^\circ(\bar{\mathbf{t}})} \prod_{u,j\colon u \overset{j}{\not\to} v} [1-P_{u,v}]$$

in (5.4.56) should be replaced by

$$\prod_{u,j\colon u \overset{j}{\not\to} v} \Big[1 - \sum_{v\in V^\circ(\bar{\mathbf{t}})} P_{u,v}\Big].$$

This creates some minor changes in what follows.

18 Page 222: (5.4.60) should read

$$\sup_{s\in[\varepsilon,1]} \Big| \frac{1}{n} \sum_{u\in(sn,n]} 1/S_u^{(n)} - \int_s^1 t^{-\chi} \mathrm{d}t \Big| \xrightarrow{\mathbb{P}} 0. \tag{1.1.5}$$

[Thanks to Haodong Zhu for noting this typo.]

19 Page 223: (5.4.64) should read

$$\prod_{u,j\colon u \overset{j}{\not\to} v} \Big[1 - \sum_{v\in V^\circ(\bar{\mathbf{t}})} P_{u,v}\Big] = (1+o_{\mathbb{P}}(1)) \prod_{v\in V^\circ(\bar{\mathbf{t}})} \mathrm{e}^{-(2m+\delta)(v\psi_v)\kappa(v/n)}. \tag{1.1.6}$$

[Thanks to Haodong Zhu for noting this typo.]

20 Page 224: In the paragraph below (5.4.69), it should read 'while $q_s' \leq m|V(\bar{\mathbf{t}})|$ is uniformly bounded.' [Thanks to Haodong Zhu for noting this typo.]

21 Page 336: Figures 8.1(a) and (b) should be reversed.

22 Page 345, line -8: $\varepsilon\pi$ should be $\vec{\pi}$.

23 Page 350: This is a more involved correction. Equation (8.5.37) should read

$$\begin{aligned}\sum_{s=1}^{n} \sum_{i=0}^{q_s-1} -\frac{\alpha}{\alpha+\beta_s+i} &= \sum_{s=1}^{n} \sum_{i=0}^{q_s-1} \Big(-\frac{m+\delta}{(2m+\delta)s} + O(q_s/s^2)\Big) \\ &= O(1)\sum_{s=1}^{n} \frac{q_s^2}{s^2} - \gamma \sum_{s=1}^{n} \frac{q_s}{s}. \end{aligned} \tag{1.1.7}$$

With this change, the bounds no longer work, and below is a novel proof.

The starting point is (8.5.31), which states that

$$\prod_{s=2}^{n}\frac{(\beta_s+q_s-1)_{q_s}}{(\alpha+\beta_s+p_s+q_s-1)_{p_s+q_s}} = \prod_{s=2}^{n}\frac{1}{(\alpha+\beta_s+p_s+q_s-1)_{p_s}}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big). \tag{1.1.8}$$

Combining (8.5.33) and (**??**)8.5.34), the first factor in (1.1.8) is bounded by

$$\prod_{s=2}^{n}\frac{1}{(\alpha+\beta_s+p_s+q_s-1)_{p_s}} \leq (1+k/\pi_{\min})^{O(1)}(2m+\delta)^{-k}\prod_{i=1}^{k}\frac{1}{(\pi_{i-1}\wedge\pi_i)}. \tag{1.1.9}$$

Further, in (8.5.41), the second factor in (1.1.8) is bounded by

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big) = \Big(1+\frac{k}{\pi_{\min}}\Big)^{O(1)}\prod_{i=1}^{k}\frac{(\pi_{i-1}\wedge\pi_i)^{\gamma}}{(\pi_{i-1}\vee\pi_i)^{\gamma}}. \tag{1.1.10}$$

Thus, in order to correct the bound, it suffices to prove that

$$\prod_{s=2}^{n}\frac{(\beta_s+q_s-1)_{q_s}}{(\alpha+\beta_s+p_s+q_s-1)_{p_s+q_s}} \leq (1+k/\pi_{\min})^{O(1)}(2m+\delta)^{-k}\prod_{i=1}^{k}\frac{1}{(\pi_{i-1}\wedge\pi_i)^{1-\gamma}(\pi_{i-1}\vee\pi_i)^{\gamma}}. \tag{1.1.11}$$

That is what we do now, bounding the two factors on the right-hand side of (1.1.8) separately.

We can write the first factor in (1.1.8) as

$$\prod_{s=2}^{n}\frac{1}{(\alpha+\beta_s+p_s+q_s-1)_{p_s}} = \prod_{s=2}^{n}\frac{\Gamma(\alpha+\beta_s+q_s)}{\Gamma(\alpha+\beta_s+p_s+q_s)} \leq \prod_{s=2}^{n}\frac{1}{(\alpha+\beta_s+q_s)^{p_s}}. \tag{1.1.12}$$

Recall that $p_s=\sum_{i=0}^{k-1}\mathbb{1}_{\{\pi_i=s\}}[\mathbb{1}_{\{\pi_{i-1}>s\}}+\mathbb{1}_{\{\pi_{i+1}>s\}}]$, so that

$$\begin{aligned}\prod_{s=2}^{n}&\frac{1}{(\alpha+\beta_s+p_s+q_s-1)_{p_s}}\\ &\leq \prod_{i=0}^{k-1}\prod_{s=2}^{n}(\alpha+\beta_s+q_s)^{-\mathbb{1}_{\{\pi_i=s\}}[\mathbb{1}_{\{\pi_{i-1}>s\}}+\mathbb{1}_{\{\pi_{i+1}>s\}}]}\\ &= \prod_{i=0}^{k-1}(\alpha+\beta_{\pi_i}+q_{\pi_i})^{-[\mathbb{1}_{\{\pi_{i-1}>\pi_i\}}+\mathbb{1}_{\{\pi_{i+1}>\pi_i\}}]}\\ &= \prod_{i=1}^{k}\frac{1}{\alpha+\beta_{(\pi_i\wedge\pi_{i-1})}+q_{(\pi_i\wedge\pi_{i-1})}}.\end{aligned} \tag{1.1.13}$$

This bounds the first term in (1.1.8).

We note that the second factor in (1.1.8), which equals (8.5.36), can be written as

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big) = \prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\frac{\beta_s+i}{\alpha+\beta_s+i} = \prod_{s=1}^{n}\frac{\Gamma(\beta_s+q_s)\Gamma(\alpha+\beta_s)}{\Gamma(\beta_s)\Gamma(\alpha+\beta_s+q_s)}. \tag{1.1.14}$$

Thus, in order to correct the argument, we need to identify the asymptotics of

$$\prod_{s=1}^{n}\frac{\Gamma(\beta_s+q_s)\Gamma(\alpha+\beta_s)}{\Gamma(\beta_s)\Gamma(\alpha+\beta_s+q_s)}. \tag{1.1.15}$$

We can use Stirling to approximate; see also [2],

$$\frac{\Gamma(t+a)}{\Gamma(t)} = t^a\Big[1 + \frac{a(a-1)}{2t} + O(t^{-2})\Big].$$

Therefore,

$$\frac{\Gamma(\beta_s+q_s)\Gamma(\alpha+\beta_s)}{\Gamma(\beta_s)\Gamma(\alpha+\beta_s+q_s)} = \frac{\beta_s^\alpha}{(\beta_s+q_s)^\alpha}\Big[1 + \tfrac{1}{2}\alpha(\alpha-1)\big(\frac{1}{\beta_s} - \frac{1}{\beta_s+q_s}\big) + O(\beta_s^{-2})\Big].$$

Bounding

$$\frac{1}{\beta_s} - \frac{1}{\beta_s+q_s} \le \frac{q_s}{\beta_s^2},$$

we can deal with this term as before. As a result, we obtain

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1 - \frac{\alpha}{\alpha+\beta_s+i}\Big) \le C(1+k/\pi_{\min})^{O(1)}\prod_{s=1}^{n}\Big(\frac{\beta_s}{\beta_s+q_s}\Big)^\alpha. \tag{1.1.16}$$

Let $\mathcal{S} = \{s_1, \dots, s_\ell\}$ denote the set of s for which $q_s \neq q_{s-1}$. Then,

$$\begin{aligned}\prod_{s=1}^{n}\frac{\beta_s}{\beta_s+q_s} &= \prod_{l}\prod_{s=s_{l-1}+1}^{s_l}\frac{\beta_s}{\beta_s+q_{s_l}} \\ &= \prod_{l}\prod_{s=s_{l-1}+1}^{s_l}\frac{s-1-m/(2m+\delta)}{s-1+q_{s_l}/(2m+\delta)-m/(2m+\delta)} \\ &= \frac{\Gamma(2-m/(2m+\delta))}{\Gamma(n-m/(2m+\delta))}\prod_{l}\frac{\Gamma(s_l+q_{s_l}/(2m+\delta)-m/(2m+\delta))}{\Gamma(s_{l-1}+q_{s_l}/(2m+\delta)-m/(2m+\delta))} \\ &= \frac{\Gamma(2-m/(2m+\delta))}{\Gamma(n-m/(2m+\delta))}\prod_{l}\frac{\Gamma(s_l+q'_{s_l}-m/(2m+\delta))}{\Gamma(s_{l-1}+q'_{s_l}-m/(2m+\delta))},\end{aligned} \tag{1.1.17}$$

where

$$q'_s = \frac{q_s}{2m+\delta}. \tag{1.1.18}$$

We then write

$$\begin{aligned}&\prod_{l}\frac{\Gamma(s_l+q'_{s_l}-m/(2m+\delta))}{\Gamma(s_i+q'_{s_{l-1}}-m/(2m+\delta))} \\ &= \prod_{l}\frac{\Gamma(s_l+q'_{s_l}-m/(2m+\delta))}{\Gamma(s_{l-1}+q'_{s_{l-1}}-m/(2m+\delta))}\prod_{l}\frac{\Gamma(s_l+q'_{s_{l-1}}-m/(2m+\delta))}{\Gamma(s_{l-1}+q'_{s_l}-m/(2m+\delta))} \\ &= \frac{\Gamma(n+q'_n-m/(2m+\delta))}{\Gamma(3+q'_3-m/(2m+\delta))}\prod_{l}\frac{\Gamma(s_{l-1}+q'_{s_{l-1}}-m/(2m+\delta))}{\Gamma(s_{l-1}+q'_{s_l}-m/(2m+\delta))} \\ &= \frac{\Gamma(n-m/(2m+\delta))}{\Gamma(2-m/(2m+\delta))}\prod_{l}\frac{\Gamma(s_{l-1}+q'_{s_{l-1}}-m/(2m+\delta))}{\Gamma(s_{l-1}+q'_{s_l}-m/(2m+\delta))},\end{aligned} \tag{1.1.19}$$

since $q_n = q_2 = 0$. Thus,

$$\prod_{s=1}^{n} \frac{\beta_s}{\beta_s + q_s} = \prod_l \frac{\Gamma(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta))}{\Gamma(s_{l-1} + q'_{s_l} - m/(2m+\delta))}$$
$$= \prod_l \frac{\Gamma(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta))}{\Gamma(s_{l-1} + q'_{s_l} - m/(2m+\delta))}$$
$$= \prod_l \frac{\Gamma(s_{l-1} + q'_{s_i} - \Delta q'_{s_l} - m/(2m+\delta))}{\Gamma(s_{l-1} + q'_{s_l} - m/(2m+\delta))}, \tag{1.1.20}$$

where we use that

$$\Delta q'_s = q'_s - q'_{s-1} \in \{-2, -1, 1, 2\}/(2m+\delta),$$

and $\Delta q'_{s_l} = q'_{s_l} - q'_{s_l - 1} = q'_{s_l} - q'_{s_{l-1}}$.

Note that q_s only changes when s is close to $\pi_i \wedge \pi_{i+1}$ or $\pi_i \vee \pi_{i+1}$ for some i. More precisely, using that

$$q_s = \sum_{i=0}^{k-1} \mathbb{1}_{\{s \in (\pi_i, \pi_{i+1}) \cup (\pi_{i+1}, \pi_i)\}} = \sum_{i=0}^{k-1} \mathbb{1}_{\{s \in (\pi_i \wedge \pi_{i+1}, \pi_{i+1} \vee \pi_i)\}},$$

we obtain

$$\Delta q_s = q_s - q_{s-1} = \sum_{i=0}^{k-1} [\mathbb{1}_{\{s \in (\pi_i \wedge \pi_{i+1}, \pi_{i+1} \vee \pi_i)\}} - \mathbb{1}_{\{s-1 \in (\pi_i \wedge \pi_{i+1}, \pi_{i+1} \vee \pi_i)\}}$$
$$= \sum_{i=0}^{k-1} [\mathbb{1}_{\{s = (\pi_i \wedge \pi_{i+1})+1\}} - \mathbb{1}_{\{s = (\pi_{i+1} \vee \pi_i)\}}], \tag{1.1.21}$$

which indeed is in $\{-2, -1, 1, 2\}$ since $\vec{\pi}$ is self-avoiding. Therefore,

$$\prod_{s=1}^{n} \frac{\beta_s}{\beta_s + q_s} = \prod_l \frac{\Gamma(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta))}{\Gamma(s_{l-1} + q'_{s_l} - m/(2m+\delta))}$$
$$= \prod_l \frac{\Gamma(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta))}{\Gamma(s_{l-1} + q'_{s_{l-1}} + \Delta q'_{s_l} - m/(2m+\delta))}$$
$$= \prod_l \Big(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta)\Big)^{-\Delta q'_{s_l}} \Big[1 + \frac{\Delta q'_{s_l}(\Delta q'_{s_l} - 1)}{s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta)} + O(s_{l-1}^{-2})\Big]$$
$$\leq C(1 + k/\pi_{\min})^{O(1)} \prod_l \Big(s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta)\Big)^{-\Delta q'_{s_l}}, \tag{1.1.22}$$

since

$$\prod_l \Big[1 + \frac{\Delta q'_{s_l}(\Delta q'_{s_l} - 1)}{s_{l-1} + q'_{s_{l-1}} - m/(2m+\delta)} + O(s_{l-1}^{-2})\Big]$$
$$= \prod_{s=1}^{n} \Big[1 + \frac{\Delta q'_s(\Delta q'_s - 1)}{s + q'_s - m/(2m+\delta)} + O(s^{-2})\Big] \leq C(1 + k/\pi_{\min})^{O(1)}. \tag{1.1.23}$$

This follows since, by (1.1.21), $\Delta q'_s = 0$ unless $s = \pi_i$ or $s = \pi_i + 1$ for some i, and thus

$$\prod_{s=1}^{n} \Big[1 + \frac{\Delta q'_s(\Delta q'_s - 1)}{s + q'_s - m/(2m+\delta)} + O(s^{-2})\Big]$$
$$\leq C \prod_{i=1}^{k} [1 + \frac{2}{\pi_i}][1 + \frac{2}{\pi_i + 1}] \leq C \prod_{i=1}^{k} [1 + \frac{2}{\pi_{\min} + i}]^2 \leq C(1 + k/\pi_{\min})^{O(1)}, \tag{1.1.24}$$

since $\vec{\pi}$ is self-avoiding.

Thus, using that $\alpha/(2m+\delta)=\gamma$, the second factor in (1.1.8) can be bounded by

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big)\leq Ck^p\prod_{s=1}^{n}\Big(\frac{\beta_s}{\beta_s+q_s}\Big)^{\alpha}$$
$$\leq C(1+k/\pi_{\min})^{O(1)}\prod_{i}\Big(s_{l-1}+q'_{s_{l-1}}-m/(2m+\delta)\Big)^{-\gamma\Delta q_{s_l}}. \tag{1.1.25}$$

Since $\sum_l \Delta q_{s_l}=0$, we can multiply each term by $2m+\delta$, to obtain

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big)\leq C(1+k/\pi_{\min})^{O(1)}\prod_{l}\Big(\beta_{s_{l-1}}+q_{s_{l-1}}\Big)^{-\gamma\Delta q_{s_l}}. \tag{1.1.26}$$

By (1.1.21),

$$\prod_{s=1}^{n}\prod_{i=0}^{q_s-1}\Big(1-\frac{\alpha}{\alpha+\beta_s+i}\Big)\leq Ck^a\prod_{s=1}^{n}\Big(\frac{\beta_s}{\beta_s+q_s}\Big)^{\alpha}$$
$$\leq C(1+k/\pi_{\min})^{O(1)}\prod_{i}\prod_{l}\frac{(\beta_{s_{l-1}}+q_{s_{l-1}})^{\gamma\mathbb{1}\{s_{l-1}=(\pi_i\wedge\pi_{i+1})+1\}}}{(\beta_{s_{l-1}}+q_{s_{l-1}})^{\gamma\mathbb{1}\{s_{l-1}=(\pi_{i+1}\vee\pi_i)\}}}$$
$$=C(1+k/\pi_{\min})^{O(1)}\prod_{i}\frac{(\beta_{(\pi_i\wedge\pi_{i+1})+1}+q_{(\pi_i\wedge\pi_{i+1})+1})^{\gamma}}{(\beta_{(\pi_{i+1}\vee\pi_i)}+q_{(\pi_{i+1}\vee\pi_i)})^{\gamma}}. \tag{1.1.27}$$

We conclude that (1.1.8) can be bounded by

$$\prod_{s=2}^{n}\frac{(\beta_s+q_s-1)_{q_s}}{(\alpha+\beta_s+p_s+q_s-1)_{p_s+q_s}}$$
$$\leq C(1+k/\pi_{\min})^{O(1)}\prod_{i=1}^{k}\frac{(\beta_{(\pi_i\wedge\pi_{i+1})+1}+q_{(\pi_i\wedge\pi_{i+1})+1})^{\gamma}}{(\alpha+\beta_{(\pi_i\wedge\pi_{i-1})}+q_{(\pi_i\wedge\pi_{i-1})})(\beta_{(\pi_{i+1}\vee\pi_i)}+q_{(\pi_{i+1}\vee\pi_i)})^{\gamma}}$$
$$\leq C(1+k/\pi_{\min})^{O(1)}\prod_{i=1}^{k}\frac{1}{(\alpha+\beta_{(\pi_i\wedge\pi_{i-1})}+q_{(\pi_i\wedge\pi_{i-1})})^{1-\gamma}(\beta_{(\pi_{i+1}\vee\pi_i)}+q_{(\pi_{i+1}\vee\pi_i)})^{\gamma}}, \tag{1.1.28}$$

where the last bound is due to the fact that $\alpha=m+\delta\geq 1$ when $m\geq 1$ and $\delta>0$. Thus, using that $\alpha,q_s\geq 0$,

$$\prod_{s=2}^{n}\frac{(\beta_s+q_s-1)_{q_s}}{(\alpha+\beta_s+p_s+q_s-1)_{p_s+q_s}}$$
$$\leq C(1+k/\pi_{\min})^{O(1)}\prod_{i=1}^{k}\frac{1}{\alpha+\beta^{1-\gamma}_{(\pi_i\wedge\pi_{i-1})}\beta^{\gamma}_{(\pi_{i+1}\vee\pi_i)}}, \tag{1.1.29}$$

and

$$q_s=(2m+\delta)s-(3m+\delta), \tag{1.1.30}$$

and, for all $b<\pi_{\min}$,

$$\prod_{i=1}^{k}\frac{1}{(1-\frac{b}{\pi_i\wedge\pi_{i-1}})(1-\frac{b}{\pi_i\vee\pi_{i-1}})}\leq(1+k/\pi_{\min})^{O(1)}, \tag{1.1.31}$$

we arrive at

$$\prod_{s=2}^{n} \frac{(\beta_s + q_s - 1)_{q_s}}{(\alpha + \beta_s + p_s + q_s - 1)_{p_s+q_s}} \le C(1 + k/\pi_{\min})^{O(1)}(2m + \delta)^{-k} \prod_{i=1}^{k} \frac{1}{(\pi_i \wedge \pi_{i-1})^{1-\gamma}(\pi_{i+1} \vee \pi_i)^{\gamma}}, \tag{1.1.32}$$

as required in (1.1.11).

[Thanks to Haodong Zhu for noting this typo, and commenting on its solution.]

REFERENCES

[1] R. van der Hofstad. *Random graphs and complex networks. Volume 2*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2024.

[2] F. G. Tricomi and A. Erdélyi. The asymptotic expansion of a ratio of gamma functions. *Pacific J. Math.*